

On triviality of some reduced Whitehead groups over Henselian fields

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Abstract

Let F be a Henselian field of q -cohomological dimension 3, where q is a prime and let Γ_F be the totally ordered abelian value group of F . Let D be a central division algebra over F of q -primary index such that the characteristic of the residue field \overline{F} , $\text{char}(\overline{F})$ is coprime to q . We show that when $cd_q(\overline{F}) < \infty$ and $1 \leq \dim_{\mathbb{F}_q}(\Gamma_F/q\Gamma_F) \leq 3$, the reduced Whitehead group of D is trivial. We also show that if $cd_q(\overline{F}) < \infty$ and $\dim_{\mathbb{F}_q}(\Gamma_F/q\Gamma_F) = 0$ then, the reduced Whitehead group of D is trivial when D is semiramified or totally ramified.

1 Introduction

Let E be an arbitrary field and A be a finite-dimensional central simple algebra over E . We denote the group of units of A by A^* . The *reduced Whitehead group* of A is given by,

$$SK_1(A) = \{a \in A^* : \text{Nrd}_A(a) = 1\} / [A^* : A^*],$$

where $[A^* : A^*]$ is the commutator subgroup of A^* and Nrd_A is the reduced norm map, $\text{Nrd}_A : A^* \rightarrow E^*$. Let D be an underlying division algebra of A , i.e., $A \simeq M_n(D)$ for some natural number n . By (§23, corollary 1, [D]), there is an isomorphism $SK_1(A) \simeq SK_1(D)$. Moreover, if D_i ($1 \leq i \leq r$) are central division algebras of p_i -primary degrees, where p_i 's are prime then, $SK_1(D_1 \otimes D_2 \otimes \cdots \otimes D_r) \simeq SK_1(D_1) \times SK_1(D_2) \times \cdots \times SK_1(D_r)$ (see §23, lemma 6, [D]). Thus, in order to study $SK_1(A)$ it is enough to study central division algebras of prime power degree.

Let G_E denote the absolute Galois group of a field E , i.e., $G_E = \text{Gal}(E_s/E)$, where E_s is the separable closure of E . Recall that the q -cohomological dimension of E is the least positive integer d such that for all discrete G_E -modules A which are q -primary torsion groups, the Galois cohomology groups $H^i(G_E, A)$ are trivial for $i \geq d + 1$. The q -cohomological dimension of E is denoted by $cd_q(E)$.

In this article our aim is to prove the following theorem.

Theorem 1.1 *Let (F, v) be a Henselian field with totally ordered abelian value group Γ_F and the characteristic of the residue field \overline{F} , $\text{char}(\overline{F}) = \bar{p}$. Let $q \neq \bar{p}$ be a prime and D be a finite-dimensional division algebra over F with center $Z(D) = F$. Let the degree of D , $\deg(D) = q^s$ for some natural number s . We denote the q -rank of Γ_F by $r_q = \dim_{\mathbb{F}_q}(\Gamma_F/q\Gamma_F)$. Let the q -cohomological dimension of \overline{F} , $cd_q(\overline{F}) < \infty$ and the q -cohomological dimension of F , $cd_q(F) = 3$. Then,*

1. If $1 \leq r_q \leq 3$ then, $SK_1(D) = (1)$.

2. If $r_q = 0$ then, $SK_1(D) = (1)$ when D is semiramified or totally ramified.

In order to prove the above theorem we first obtain a relation between the q -cohomological dimension of a Henselian field and its residue field. More precisely, we prove the following proposition.

Proposition 1.2 *Let (F, v) be a Henselian field with totally ordered abelian value group Γ_F and the characteristic of residue field, $\text{char}(\overline{F}) = \bar{p}$. For a prime $q \neq \bar{p}$, let the q -rank of Γ_F be finite, i.e., $r_q = \dim_{\mathbb{F}_q}(\Gamma_F/q\Gamma_F) < \infty$. If the q -cohomological dimension of \overline{F} , $cd_q(\overline{F}) < \infty$ then, we have*

$$cd_q(F) = cd_q(\overline{F}) + r_q.$$

Along with the above proposition (1.2), the proof of theorem (1.1) uses valuation theory on division algebras over Henselian fields as developed in [HW] and [TW-15]. It has been noted that computations in the graded settings are often easier and explicit (see chapter 11, [TW-15]). For terminology and notations we refer the reader to §2 below. The reduced Whitehead group of a tame division algebra over a Henselian field can be computed in terms of the information on the residue field (cf. [HW]). Indeed, for a tame division algebra D over a Henselian field we associate a graded division algebra $\text{gr}(D)$ to D . The associated graded division algebra $\text{gr}(D)$ encodes information on the residue division algebra, the value group of D , and the *canonical homomorphism* (see §1.3, [TW-15]). A tame division algebra over a Henselian field can be determined, up to isomorphism, by the associated graded division algebra (cf. corollary 8.6, [TW-15]). As in the ungraded case, the reduced Whitehead group of $\text{gr}(D)$, $SK_1(\text{gr}(D))$ can be defined (see §2.4). In [HW], authors show that $SK_1(D)$ is isomorphic to $SK_1(\text{gr}(D))$, and they compute the reduced Whitehead group of $\text{gr}(D)$, $SK_1(\text{gr}(D))$ in terms of the information on the residue field (see §3.2).

The triviality of the reduced Whitehead group was a longstanding problem. It was conjectured that $SK_1(A)$ is trivial for every central simple algebra A over a field E . In 1975, Platonov gave examples of algebras with non-trivial SK_1 using valuation theory on division algebras. Platonov constructed division algebras over iterated Laurent series field over a global field. The non-triviality of SK_1 can be used to show that the algebraic group $SL_1(D)$ is not a rational variety. Indeed, for any finite-dimensional division algebra over E , $SK_1(D) \simeq SL_1(D)/R$ where $SL_1(D)/R$ is the group of R -equivalence classes (as defined by Manin) on the algebraic group $SL_1(D)$ (see 6.§18, [V]). It is known that if the group of R -equivalence classes of an algebraic group is non-trivial then, the algebraic group is not rational. In [M], Merkurjev show that the reduced Whitehead group of A is *generically non-trivial*, i.e., there is a field extension L/F such that $SK_1(A \otimes_F L) \neq (1)$. The generic non-triviality of the reduced Whitehead group was conjectured by Suslin. On the other hand, the triviality results for the reduced Whitehead group were obtained by considering central simple algebras over fields with low cohomological dimension. In [W], it has been proved that if $\text{ind}(A)$ is square-free then, $SK_1(A) = (1)$. He also showed that if E is a number field then, $SK_1(A) = (1)$. In [Y], the triviality of SK_1 was obtained for division algebras over fields of cohomological dimension at most 2. While in [G], it is proved that if the index of A , $\text{ind}(A) = q^s$ is coprime to $\text{char}(E)$ and $cd_q(E) \leq 2$ then, $SK_1(A) = (1)$. Suslin conjectured that if $cd(E) \leq 3$ then, the reduced Whitehead group of a central simple algebra over E is trivial. The triviality of SK_1 for a central simple algebra of degree 4 over a field of cohomological dimension 3 is proved by Rost (see IV.§17.A, [KMRT]). The theorem (1.1) above only partially answers Suslin's conjecture in affirmative for certain q -primary central division algebras over Henselian fields.

2 Preliminaries

In this section we recall some notions on Henselian valued fields and valued division algebras over Henselian fields. Many of the notions and definitions can be given more generally for graded division algebras over graded rings. We refer the reader to [EP] and [TW-15] for more details.

2.1 Notations and basic definitions

We start with the definition of a graded ring. Let Γ be a torsion-free abelian group, written additively. Let A be a ring with unity 1 with direct sum decomposition $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where each A_γ is an additive abelian group and $A_\gamma \cdot A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. The elements of $\bigcup_{\gamma \in \Gamma} A_\gamma$ are called *homogeneous elements* of A . A graded ring D is called a *graded division ring* if $1 \neq 0$ in D and every nonzero homogeneous element of D is a unit. A commutative graded division ring is called a graded field. Throughout this article, division algebras (respectively, graded algebras) are always assumed to be finite-dimensional over a field (respectively, graded field).

Let (F, v) be a Henselian valued field with a valuation v . Throughout this article we fix a valuation $v : F \rightarrow \Gamma \cup \infty$, where Γ is a totally ordered additive abelian group. We denote image of the valuation v , $v(F^*)$ by Γ_F . Let D be a finite-dimensional central division algebra over F . For a central division algebra D over a Henselian valued field (F, v) , valuation v extends uniquely to the valuation v_D on D (cf. theorem 1.4, [TW-15]) and it is given by

$$v_D(d) := \frac{1}{\deg(D)} \cdot v(\text{Nrd}(d)), \text{ for } d \in D^*.$$

Associated to the valuation v_D there are following structures:

1. $\Gamma_D = v_D(D^*)$, the *value group* of v_D .
2. $\mathcal{O}_D = \{x \in D : v_D(x) \geq 0\}$, which is a subring of D called the *valuation ring* of D .
3. $\mathfrak{m}_D = \{x \in D : v_D(x) > 0\}$, which is a unique two-sided ideal of \mathcal{O}_D .
4. $\overline{D} = \mathcal{O}_D / \mathfrak{m}_D$, the residue division ring.

There is a well-defined group homomorphism (cf. §1.1.1, [TW-15])

$$\theta_D : \Gamma_D \rightarrow \text{Aut}(Z(\overline{D})/\overline{F}),$$

where \overline{F} is the residue field of F with respect to the valuation v . This group homomorphism is given as follows: for any $\gamma \in \Gamma_D$ and any $d \in D^*$ with $v_D(d) = \gamma$ and any $x \in \mathcal{O}_D$ with $\bar{x} := (x + \mathfrak{m}_D) \in Z(\overline{D})$,

$$\theta_D(\gamma)(\bar{x}) = \overline{dx d^{-1}}.$$

We call θ_D the *canonical homomorphism* of the valuation v_D on D . Since $\Gamma_F \subseteq \ker(\theta_D)$, there is an induced homomorphism

$$\bar{\theta}_D : \Gamma_D / \Gamma_F \rightarrow \text{Aut}(Z(\overline{D})/\overline{F}).$$

The valuation $v_D : D \rightarrow \Gamma \cup \infty$ defines a filtration on D : we set for each $\gamma \in \Gamma$

$$D_{\geq \gamma} = \{x \in D : v_D(x) \geq \gamma\} \text{ and } D_{> \gamma} = \{x \in D : v_D(x) > \gamma\}.$$

Note that $D_{\geq \gamma} = D_{> \gamma}$ if and only if $\gamma \notin \Gamma_D$. For each $\gamma \in \Gamma$ we set

$$D_\gamma = D_{\geq \gamma} / D_{> \gamma},$$

a left and right \overline{D} -vector space of dimension 1. For any $\gamma, \delta \in \Gamma$ we have

$$D_{\geq \gamma} D_{\geq \delta} \subseteq D_{\geq \gamma + \delta}, \quad D_{> \gamma} D_{\geq \delta} \subseteq D_{> \gamma + \delta}, \quad D_{\geq \gamma} D_{> \delta} \subseteq D_{> \gamma + \delta},$$

hence there is a well-defined multiplication operation

$$D_\gamma \times D_\delta \rightarrow D_{\gamma + \delta}$$

given by $(x + D_{> \gamma}) \cdot (y + D_{> \delta}) = (xy + D_{\gamma + \delta})$, for all $x \in D_{\geq \gamma}$ and $y \in D_{\geq \delta}$. We define the *associated graded ring of v_D on D* to be

$$\mathbf{gr}(D) = \bigoplus_{\gamma \in \Gamma} D_\gamma.$$

The *grade group* of $\mathbf{gr}(D)$ is defined to be

$$\Gamma_{\mathbf{gr}(D)} = \{\gamma \in \Gamma : D_\gamma \neq 0\}.$$

It is clear that $\Gamma_{\mathbf{gr}(D)} = \Gamma_D$ and $\overline{D} = \mathbf{gr}(D)_0$. The *homogeneous elements* of $\mathbf{gr}(D)$ are the elements of $\cup_{\gamma \in \Gamma} D_\gamma$. The multiplicative property of the valuation v_D shows that $\mathbf{gr}(D)$ is a *graded division ring*, i.e., every nonzero homogeneous element of $\mathbf{gr}(D)$ is a unit. For $x \in D^*$, we write \tilde{x} for the image $x + D_{> v_D(x)}$ of x in $D_{v_D(x)}$. Thus, $\tilde{x} = \bar{x}$ if $v_D(x) = 0$.

We call commutative graded division ring *graded field*. For instance, $\mathbf{gr}(F)$ is a graded field. Note that such a ring is an integral domain. A graded division ring $\mathbf{gr}(D)$ over a graded field $\mathbf{gr}(F)$ is graded free, i.e., $\mathbf{gr}(D)$ has a basis as a free $\mathbf{gr}(F)$ -module consisting of homogeneous elements (cf. proposition 2.5, [TW-15]). We write $[\mathbf{gr}(D) : \mathbf{gr}(F)]$ for the dimension of $\mathbf{gr}(F)$ -vector space $\mathbf{gr}(D)$. We then have

$$[\mathbf{gr}(D) : \mathbf{gr}(F)] = [\overline{D} : \overline{F}] \cdot |\Gamma_D : \Gamma_F|,$$

where $[\overline{D} : \overline{F}]$ denotes the dimension of \overline{F} -vector space \overline{D} and $|\Gamma_D : \Gamma_F|$ is the index in the abelian group Γ_D of its subgroup Γ_F . We have

$$[\mathbf{gr}(D) : \mathbf{gr}(F)] = [\overline{D} : \overline{F}] \cdot |\Gamma_D : \Gamma_F| \leq [D : F] < \infty. \quad (1)$$

The above inequality is called the *fundamental inequality*. We call the division algebra D *defectless* if there is an equality in the above equation (1), i.e.,

$$[D : F] = [\overline{D} : \overline{F}] \cdot |\Gamma_D : \Gamma_F|.$$

Let D be a division algebra (not necessarily central) over a Henselian field F . Note that the valuation v_D extends uniquely to the center $Z(D)$. We define the defect $\partial_{D/F}$ of D over F by

$$\partial_{D/F} = [D : F] / ([\overline{D} : \overline{F}] \cdot |\Gamma_D : \Gamma_F|).$$

It is clear that the defect, $\partial_{D/F} \geq 0$. Moreover, by the theorem of Morandi (cf. theorem 4.3, [TW-15]) the defect is given by, $\partial_{D/F} = \text{char}(\overline{F})^l$ for some $l \in \mathbb{N}$.

2.2 Notions on valued division algebras

In this section we recall notions of tame, totally ramified and semiramified division algebras over a Henselian field. For more details we refer the reader to relevant chapters in the book ([TW-15]).

Throughout this section we assume (F, v) is a Henselian valued field and D is a finite-dimensional central division algebra over F . The unique valuation on D extending v will be denoted by v_D .

A finite-dimensional central division algebra D over a Henselian field F is said to be *tame* (over F) if D is defectless over F , the extension $Z(\overline{D})/\overline{F}$ is separable and $\text{char}(\overline{F}) \nmid |\ker(\theta_D) : \Gamma_F|$, where θ_D is the canonical homomorphism defined in §2.1. By (proposition 6.63, [TW-15]), D is tame if and only if it is split by the maximal tamely ramified extension of F .

A finite-dimensional central division algebra D over a Henselian valued field F is said to be *semiramified* if the residue algebra \overline{D} is a field and

$$[\overline{D} : \overline{F}] = |\Gamma_D : \Gamma_F| = \deg(D). \quad (2)$$

In particular, D is defectless. There exists a semiramified algebra which is not tame, see (example 8.37, [TW-15]).

A finite-dimensional central division algebra over a Henselian field F is said to be *totally ramified* over F if

$$|\Gamma_D : \Gamma_F| = [D : F], \quad (3)$$

or, equivalently, $\overline{D} = \overline{F}$ and D is defectless (cf. §7.4.1, [TW-15]).

2.3 Ordered Abelian groups

In this section we recall various notions of rank of an abelian group. We refer the reader to (Appendix, A.4, [TW-15]) and (§2.1 and §3.4, [EP]) for more details.

Rank. Let Γ be a totally ordered abelian group. Note that such a group is torsion-free. A subgroup $\Delta \leq \Gamma$ is said to be *convex* whenever $\gamma \in \Gamma$ and $\delta \in \Delta$ and $0 \leq \gamma \leq \delta$, then $\gamma \in \Delta$. The *rank* of Γ is defined to be

$$rk(\Gamma) := \text{the number of convex subgroups } \Delta \text{ of } \Gamma \text{ with } \Delta \neq \Gamma.$$

If v is a valuation on a field F , then the *rank of v* is defined to be the rank of its value group:

$$rk(v) := rk(\Gamma_{F,v}).$$

Equivalently,

$$rk(v) = \text{Krull dimension of the valuation ring } \mathcal{O}_{F,v}.$$

Rational rank. For an abelian group A , the *rational rank of A* is defined to be

$$rr(A) := \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}).$$

It is clear from the definition of rational rank that, $rr(A) = 0$ if and only if A is a torsion group. For a totally ordered abelian group Γ and its any subgroup Λ we have (see proposition 3.4.1, [EP]):

$$rk(\Gamma) \leq rk(\Lambda) + rr(\Gamma/\Lambda).$$

In particular when $\Lambda = (0)$, $rk(\Gamma) \leq rr(\Gamma)$.

Torsion rank. For a finite abelian group A we define its *torsion rank* to be

$$\begin{aligned} trk(A) &= \text{minimal cardinality of generating sets of } A \\ &= \max\{\dim_{\mathbb{F}_p}(A/pA) : p \text{ is a prime}\}. \end{aligned}$$

We have following proposition:

Proposition 2.1 (*proposition A.36, [TW-15]*) *Let Γ be any torsion free abelian group, and let Λ be a subgroup of Γ with $|\Gamma : \Lambda| < \infty$. Then,*

$$trk(\Gamma/\Lambda) \leq rr(\Lambda) = rr(\Gamma).$$

2.4 Reduced Norm on Graded Division Algebras

In this section we give definition of the reduced norm map on a graded division algebra.

Let F be a graded field, i.e., F is a commutative graded division ring with unity. It is easy to see that F is an integral domain. We denote its quotient field by $q(F)$. Let A be a simple central graded algebra with center F , i.e., it is a graded ring with center F and its only two-sided homogeneous ideals are (0) and A . We denote the underlying ungraded ring of A (respectively, F) by A^\natural (respectively, F^\natural). We consider an $q(F)$ -algebra $q(A) = A^\natural \otimes_{F^\natural} q(F)$. Then for $a \in A$, we define *reduced characteristic polynomial of a* , $Prd_{A,a}$ to be the reduced characteristic polynomial $Prd_{q(A),a} \in F[X]$. Likewise we set, the reduced trace of a , $Trd_A(a) = Trd_{q(A)}(a)$ and the reduced norm of a , $Nrd_A(a) = Nrd_{q(A)}(a) \in F$. We list few properties of reduced norm:

Proposition 2.2 (*proposition 11.6, proposition 11.7, [TW-15]*) *Let A be a simple graded algebra with center F . Then,*

1. $Nrd_A(ab) = Nrd_A(a) \cdot Nrd_A(b)$ for all $a, b \in A$.
2. $Nrd_A(a) \neq 0$ if and only if $a \in A^*$.
3. Suppose $a \in A$ is homogeneous of degree δ . Then $Trd_A(a) \in F_\delta$ and $Nrd_A(a) \in F_{n\delta}$ where $n = \deg(A)$.

For more details on reduced norm and reduced trace of simple graded algebra we refer to (§11.1, [TW-15]). We now consider reduced Whitehead group for graded division algebras. Let F be a graded field and let D be a graded division algebra with center F . By the above proposition $Nrd_D : D^* \rightarrow F^*$ is a group homomorphism. Let

$$D^{(1)} = \ker(Nrd|_{D^*}) = \{a \in D^* : Nrd_D(a) = 1\}.$$

Since every element of D^* is homogeneous, $D^{(1)} \subseteq D_0^*$. Let $[D^* : D^*]$ be the commutator of D^* . Because Nrd_D is multiplicative and its image is commutative, we have $[D^* : D^*] \subseteq D^{(1)}$. Define the reduced Whitehead group of D ,

$$SK_1(D) = D^{(1)} / [D^* : D^*].$$

3 Some known results

In this section we recall some known results which are used in the proofs of main theorem. We refer to the earlier section for notation and terminology. We start with the following result.

Theorem 3.1 (theorem 4.8, [HW] or theorem 11.21, [TW-15]) *Let D be a division algebra tame over its Henselian center F . Then, there is an isomorphism*

$$SK_1(D) \simeq SK_1(\text{gr}(D)).$$

For a graded division algebra D with graded center F we have the following result analyzing reduced Whitehead group of D in terms of pieces of the graded structure on D .

Theorem 3.2 (theorem 3.4, [HW] or theorem 11.10, [TW-15]) *Let D be a graded division algebra with center F , and let $G = \text{Gal}(Z(D_0)/F_0)$. Set $\zeta = \text{ind}(D)/(\text{ind}(D_0) \cdot [Z(D_0) : F_0])$. Let $\tilde{N} = N_{Z(D_0)/F_0} \circ \text{Nrd}_{D_0} : D_0^* \rightarrow F_0^*$ and*

$$\mathcal{K} = (D_0^{(1)} \cap [D_0^*, D^*])/[D_0^*, D^*].$$

Then, the following diagram has exact rows and column:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & & \downarrow & & & \\
 \mathcal{K} & \longrightarrow & SK_1(D_0) & \longrightarrow & \ker(\tilde{N})/[D_0^*, D^*] & \xrightarrow{\text{Nrd}_{D_0}} & \hat{H}^{-1}(G, \text{Nrd}_{D_0}(D_0^*)) \longrightarrow 1 \\
 & & & \downarrow & & & \\
 & & \Gamma_D/\Gamma_F \wedge \Gamma_D/\Gamma_F & \xrightarrow{\kappa} & D^{(1)}/[D_0^*, D^*] & \longrightarrow & SK_1(D) \longrightarrow 1 \\
 & & & & \downarrow \tilde{N} & & \\
 & & & & \mu_\zeta(F_0) \cap \tilde{N}(D_0^*) & & \\
 & & & & \downarrow & & \\
 & & & & 1 & &
 \end{array}$$

The map κ in the diagram is given as follows: For $\gamma, \delta \in \Gamma_D$ choose any non-zero $d_\gamma \in D_\gamma$ and $d_\delta \in D_\delta$. Then

$$\kappa((\gamma + \Gamma_F) \wedge (\delta + \Gamma_F)) = [d_\gamma, d_\delta] [D_0^*, D^*].$$

Now we state a result on the triviality of reduced Whitehead group of q -primary division algebra over a field K of q -cohomological dimension at most 2. More precisely we have:

Theorem 3.3 (theorem 1.1, [G]) *Let K be a field of characteristic p (which can be zero) and let q be a prime number different from p . Suppose that $cd_q(K) \leq 2$ and that A is a central simple algebra over K whose index is q -primary. Then, the reduced Whitehead group $SK_1(A)$ is trivial, i.e., $SK_1(A) = (1)$.*

4 Computation of cohomological dimension

Let F be a Henselian field with valuation v . Let Γ_F and \overline{F} be the value group and the residue field of F with respect to v . We assume that $\text{char}(\overline{F}) = \overline{p}$. We denote by F^t the inertia field of F and $G^t = \text{Gal}(F^s/F^t)$ its corresponding Galois group (see §5.2, [EP]). We denote by G_F the absolute Galois group of the field F . By (theorem 5.2.7 (3), [EP]) the absolute Galois group of the residue field \overline{F} , $G_{\overline{F}} = \text{Gal}(F^t/F) = G_F/G^t$. Consider the inflation map induced by the above isomorphism:

$$\text{inf} : H^i(G_{\overline{F}}, \mathbb{Z}/n) \rightarrow H^i(G_F, \mathbb{Z}/n); \text{ for } (n, \overline{p}) = 1.$$

Let q be a prime distinct from \overline{p} . By (theorem 5.3.3 (1), [EP]), the ramification Galois group $G^v = \text{Gal}(F^s/F^v)$ is a pro- \overline{p} subgroup of G^t . Hence $H^i(G^v, \mathbb{Z}/n) = 0$ for $i \geq 1$, and for n coprime to \overline{p} (cf. I.§3.3, corollary 2, [Se2]). Thus by the inflation-restriction sequence (VII.§6, proposition 5, [Se1]) we get the following isomorphism

$$H^i(G^t/G^v, \mathbb{Z}/n) \simeq H^i(G^t, \mathbb{Z}/n), \text{ for } i \geq 0 \text{ and } (n, \overline{p}) = 1.$$

By (theorem 5.3.3 (3), [EP]), there is an isomorphism of profinite groups

$$G^t/G^v \simeq \prod_{q \neq \overline{p}} \mathbb{Z}_q^{r_q} \quad (4)$$

where for each prime $q \neq \overline{p}$, r_q is the q -rank of Γ_F , i.e., r_q is the \mathbb{F}_q -dimension of $\Gamma_F/q\Gamma_F$. Thus

$$H^i(G^t, \mathbb{Z}/n) \simeq H^i(G^t/G^v, \mathbb{Z}/n) \simeq H^i\left(\prod_{q \neq \overline{p}} \mathbb{Z}_q^{r_q}, \mathbb{Z}/n\right); \text{ for } i \geq 0 \text{ and } (n, \overline{p}) = 1. \quad (5)$$

We first prove the following lemma which appear as a part of an exercise in [Se2].

Lemma 4.1 (I.§4.5, Exercise (1), [Se2]) *For a prime number q and a natural number r_q we have,*

$$cd_q(\mathbb{Z}_q^{r_q}) = r_q \text{ and } H^{r_q}(\mathbb{Z}_q^{r_q}, \mathbb{Z}/q) \text{ is a group of order } q.$$

Proof : We prove this lemma by induction on r_q . Suppose $r_q = 1$. Then for any prime q , $cd_q(\mathbb{Z}_q) = 1$ and $H^1(\mathbb{Z}_q, \mathbb{Z}/q)$ is a group of order q . We assume the result for $r_q - 1 < \infty$, i.e., $cd_q(\mathbb{Z}_q^{r_q-1}) = r_q - 1$ and $H^{r_q-1}(\mathbb{Z}_q^{r_q-1}, \mathbb{Z}/q)$ is a group of order q . Now to prove the result for r_q , consider the short exact sequence

$$1 \rightarrow \mathbb{Z}_q \rightarrow \mathbb{Z}_q^{r_q} \rightarrow \mathbb{Z}_q^{r_q-1} \rightarrow 1.$$

Note that, by our induction hypothesis $cd_q(\mathbb{Z}_q^{r_q-1}) = cd_q(\mathbb{Z}_q^{r_q}/\mathbb{Z}_q) = r_q - 1 < \infty$ and $cd_q(\mathbb{Z}_q) = 1$. Moreover, \mathbb{Z}_q is a pro- q -group and $H^1(\mathbb{Z}_q, \mathbb{Z}/q)$ is a group of order q . Therefore, by (I.§4, proposition 22, [Se2])

$$cd_q(\mathbb{Z}_q^{r_q}) = r_q.$$

By the spectral sequence (I.§3.3, remark, [Se2]),

$$H^{r_q}(\mathbb{Z}_q^{r_q}, \mathbb{Z}/q) = H^{r_q-1}(\mathbb{Z}_q^{r_q-1}, H^1(\mathbb{Z}_q, \mathbb{Z}/q)) = H^{r_q-1}(\mathbb{Z}_q^{r_q-1}, \mathbb{Z}/q).$$

which is a group of order q by the induction hypothesis. Hence lemma is proved. \blacksquare

We now record a consequence of the above lemma (4.1) for future reference. The proof is on the similar lines as the discrete valued case (cf. II.§4.3, proposition 12, [Se2]) and we present it here for the completeness.

Proposition 4.2 *Let F be a Henselian field with valuation v and the value group Γ_F . Let \overline{F} be the residue field of F of characteristic \overline{p} . For a prime $q \neq \overline{p}$, let the q -rank of Γ_F be finite, i.e., $r_q = \dim_{\mathbb{F}_q}(\Gamma_F/q\Gamma_F) < \infty$. If $cd_q(G_{\overline{F}}) < \infty$ then, we have*

$$cd_q(G_F) = cd_q(G_{\overline{F}}) + r_q.$$

Proof : Consider the following exact sequence

$$1 \rightarrow H \rightarrow \prod_{q' \neq \overline{p}} \mathbb{Z}_{q'}^{r_{q'}} \rightarrow \mathbb{Z}_q^{r_q} \rightarrow 1.$$

where the product is taken over all primes $q' \neq \overline{p}$ and the last map is the projection on q th component. By (I.§3.3, corollary 2, [Se2]) $cd_q(H) = 0$ and by (lemma 4.1) $cd_q(\mathbb{Z}_q^{r_q}) = r_q < \infty$. Since the group $\prod_{q' \neq \overline{p}} \mathbb{Z}_{q'}^{r_{q'}}$ is abelian, using (I.§4.1, proposition 22, [Se2]) we get

$$cd_q\left(\prod_{q' \neq \overline{p}} \mathbb{Z}_{q'}^{r_{q'}}\right) = r_q. \quad (6)$$

In fact, using inflation-restriction sequence (VII.§6, proposition 5, [Se1]) we get an isomorphism

$$H^{r_q}(\mathbb{Z}_q^{r_q}, \mathbb{Z}/q) \simeq H^{r_q}\left(\prod_{q' \neq \overline{p}} \mathbb{Z}_{q'}^{r_{q'}}, \mathbb{Z}/q\right). \quad (7)$$

In particular, $H^{r_q}(\prod_{q' \neq \overline{p}} \mathbb{Z}_{q'}^{r_{q'}}, \mathbb{Z}/q)$ is a group of order q (see lemma 4.1). Hence $H^{r_q}(G^t, \mathbb{Z}/q)$ is also a group of order q (see equation 5), and applying (I.§3.3, proposition 15, [Se2]) to the groups G^t, G^v we also have $cd_q(G^t) \leq r_q$. As a result, $cd_q(G^t) = r_q$. Moreover, using equation (6) above and (I.§3.3, proposition 15, [Se2]) we have

$$cd_q(G_F) \leq cd_q(G_{\overline{F}}) + r_q.$$

Assume $cd_q(G_F/G^t) (= cd_q(G_{\overline{F}})) = n < \infty$, using the spectral sequence (I.§3.3, remark, [Se2]), we get

$$H^{n+r_q}(G_F, \mathbb{Z}/q) = H^n(G_F/G^t, H^{r_q}(G^t, \mathbb{Z}/q)).$$

Since by our assumption $cd_q(G_F/G^t) = n$ and as observed above $H^{r_q}(G^t, \mathbb{Z}/q)$ is a group of order q , we get $H^n(G_F/G^t, H^{r_q}(G^t, \mathbb{Z}/q)) \neq 0$. Hence $cd_q(G_F) = cd_q(G_{\overline{F}}) + r_q$. \blacksquare

5 Triviality of Reduced Whitehead group

In this section we give proof of theorem (1.1) mentioned in the introduction. We start with the following lemma. We denote for a field E its absolute Galois group by G_E .

Lemma 5.1 *Let L/E be a field extension of degree q^r for some $r \in \mathbb{N}$ and a prime number q . Further assume that the q -cohomological dimension of E , $cd_q(G_E) \leq 1$. Then the multiplicative $Gal(L/E)$ -module L^* is cohomologically trivial.*

Proof : We show that the cohomology groups $H^1(Gal(L/E), L^*)$ and $H^2(Gal(L/E), L^*)$ are trivial. Indeed, by Hilbert's theorem 90 (cf. X.§1, proposition 2, [Se1]), $H^1(Gal(L/E), L^*) = 0$. By (X.§4, corollary, [Se1]) the q -group, $H^2(Gal(L/E), L^*)$ injects into the Brauer group of E . Since $cd_q(G_E) \leq 1$, the q -torsion part of the Brauer group, ${}_qBr(E) = 0$. Hence $H^2(Gal(L/E), L^*) = 0$. Thus, the Tate cohomology groups with positive exponents 1 and 2 are trivial. Using (IX.§5, theorem 8, [Se1]) we get, the $Gal(L/E)$ -module L^* is cohomologically trivial, i.e., for every subgroup $H \leq Gal(L/E)$ and every integer i , $\hat{H}^i(H, L^*) = 0$. ■

Lemma 5.2 *Let $\Gamma_F \subset \Gamma_D$ be totally ordered abelian groups with Γ_F as a subgroup of Γ_D . Suppose that the cardinality of Γ_D/Γ_F , $|\Gamma_D/\Gamma_F| < \infty$. Let r_q denote the dimension of $\Gamma_F/q\Gamma_F$ over the finite field \mathbb{F}_q . Then*

1. *If $r_q = 0$ for some prime q then, $\Gamma_D = \Gamma_F$.*
2. *If $r_q = 1$ for some prime q then, Γ_D/Γ_F is cyclic.*

Proof :

As $|\Gamma_D/\Gamma_F| < \infty$, there is a finitely generated subgroup Γ_0 of Γ_D such that $\Gamma_D = \Gamma_0 + \Gamma_F$. Therefore,

$$\Gamma_D/\Gamma_F = (\Gamma_0 + \Gamma_F)/\Gamma_F \simeq \Gamma_0/(\Gamma_0 \cap \Gamma_F).$$

Thus, $\Gamma_0 \cap \Gamma_F$ is finitely generated and the \mathbb{F}_q -subspace $(\Gamma_0 \cap \Gamma_F)/q(\Gamma_0 \cap \Gamma_F)$ of $\Gamma_F/q\Gamma_F$ is of \mathbb{F}_q -dimension at most r_q . We have,

$$\begin{aligned} rr(\Gamma_0 \cap \Gamma_F) &= rk_{\mathbb{Z}}(\Gamma_0 \cap \Gamma_F) \\ &= rk_{\mathbb{Z}/q\mathbb{Z}}((\Gamma_0 \cap \Gamma_F)/q(\Gamma_0 \cap \Gamma_F)) \\ &= trk((\Gamma_0 \cap \Gamma_F)/q(\Gamma_0 \cap \Gamma_F)). \end{aligned}$$

As $rk_{\mathbb{Z}/q\mathbb{Z}}((\Gamma_0 \cap \Gamma_F)/q(\Gamma_0 \cap \Gamma_F)) \leq r_q$, we get $rr(\Gamma_0 \cap \Gamma_F) \leq r_q$. Further we have

$$\begin{aligned} trk(\Gamma_D/\Gamma_F) &= trk((\Gamma_0 + \Gamma_F)/\Gamma_F) \\ &= trk(\Gamma_0/(\Gamma_0 \cap \Gamma_F)) \\ &\leq rr(\Gamma_0 \cap \Gamma_F); \text{ (by proposition 2.1)} \\ &\leq r_q. \end{aligned}$$

It follows that if $r_q = 0$ then $\text{trk}(\Gamma_D/\Gamma_F) = 0$, and thus $\Gamma_D = \Gamma_F$; while if $r_q = 1$ then $\text{trk}(\Gamma_D/\Gamma_F) \leq 1$, and consequently the group Γ_D/Γ_F must be cyclic. \blacksquare

We now proceed to prove the result (1.1) stated in the introduction.

Theorem 5.3 *Let (F, v) be a Henselian field with totally ordered abelian value group Γ_F and the characteristic of the residue field \bar{F} , $\text{char}(\bar{F}) = \bar{p}$. Let $q \neq \bar{p}$ be a prime and D be a finite-dimensional division algebra over F with center $Z(D) = F$. Let the degree of D , $\deg(D) = q^s$ for some natural number s . We denote the q -rank of Γ_F by $r_q = \dim_{\mathbb{F}_q}(\Gamma_F/q\Gamma_F)$. Let the q -cohomological dimension of \bar{F} , $cd_q(\bar{F}) < \infty$ and the q -cohomological dimension of F , $cd_q(F) = 3$. Then,*

1. If $1 \leq r_q \leq 3$ then, $SK_1(D) = (1)$.
2. If $r_q = 0$ then, $SK_1(D) = (1)$ when D is semiramified or totally ramified.

Proof : As $cd_q(\bar{F}) < \infty$ and $q \neq \bar{p}$, by (proposition 4.2) $cd_q(F) = cd_q(\bar{F}) + r_q$. Note that the division algebra D over the Henselian field F is tame. Indeed, we have

$$[D : F] = \partial_{D/F} \cdot [\bar{D} : \bar{F}] \cdot |\Gamma_D : \Gamma_F|,$$

where $\partial_{D/F}$ is the defect and $\partial_{D/F} = \bar{p}^l$ for some $l \in \mathbb{N}$ (see §2.2 and theorem 4.3, [TW-15]). Since, $(q, \bar{p}) = 1$ and D has q -primary degree, $\bar{p} \nmid |\ker(\theta_D) : \Gamma_F|$ and the defect, $\partial_{D/F} = 1$, i.e., in particular, D is defectless. Similarly, $\bar{p} \nmid [Z(\bar{D}) : \bar{F}]$ and hence $Z(\bar{D})/\bar{F}$ is a separable field extension.

We consider two cases $r_q = 0$ and $1 \leq r_q \leq 3$ separately.

Case 1. We show that if $cd_q(F) = 3$ and $1 \leq r_q \leq 3$ then, $SK_1(D) = (1)$. If $r_q = 1$ then, $cd_q(\bar{F}) = 2$. As $Z(\bar{D})/\bar{F}$ is a finite field extension and $cd_q(\bar{F}) = 2$, we have $cd_q(Z(\bar{D})) = 2$ (see II.§4, proposition 10, [Se2]). Thus, by (theorem 1.1, [G]) $SK_1(\bar{D}) = (1)$. As the valuation v_D is tame, the field extension $Z(\bar{D})/\bar{F}$ is separable. Moreover by (proposition 1.5 (iii), [TW-15]), the field extension $Z(\bar{D})/\bar{F}$ is normal. Hence $Z(\bar{D})/\bar{F}$ is a Galois extension. As D is tame over Henselian valued field (F, v) , the canonical homomorphism defined in (§2.1), $\theta_D : \Gamma_D \rightarrow \text{Gal}(Z(\bar{D})/\bar{F})$ is surjective (see proposition 1.5, [TW-15]). Clearly we have $\Gamma_F \subseteq \ker(\theta_D) \subseteq \Gamma_D$. By (lemma 5.2), Γ_D/Γ_F is a cyclic group and hence $\Gamma_D/\ker(\theta) \simeq \text{Gal}(Z(\bar{D})/\bar{F})$ is a cyclic group as well. By Hilbert's theorem 90 (X.§1, corollary, [Se1]), $\hat{H}^{-1}(\text{Gal}(Z(\bar{D})/\bar{F}), Z(\bar{D})^*) = (1)$. As $\deg(\bar{D})$ is q -primary and $cd_q(Z(\bar{D})) \leq 2$, by Merkurjev-Suslin theorem (theorem 24.8, [S1]), the reduced norm of \bar{D} is surjective onto its center, i.e., $\text{Nrd}_{\bar{D}}(\bar{D}^*) = Z(\bar{D})^*$. Hence,

$$\hat{H}^{-1}(\text{Gal}(Z(\bar{D})/\bar{F}), \text{Nrd}_{\bar{D}}(\bar{D}^*)) = \hat{H}^{-1}(\text{Gal}(Z(\bar{D})/\bar{F}), Z(\bar{D})^*) = (1).$$

Since D is tame over F , $\zeta = 1$. Therefore the diagram (3.2) and (theorem 3.1) gives the result, $(1) = SK_1(\text{gr}(D)) \simeq SK_1(D)$.

If $1 < r_q \leq 3$ then, $cd_q(\bar{F}) \leq 1$. Thus, there are no non-trivial q -primary division algebras over \bar{F} . In particular, $SK_1(\bar{D}) = (1)$ and by (lemma 5.1) $\hat{H}^{-1}(\text{Gal}(Z(\bar{D})/\bar{F}), Z(\bar{D})^*) = (1)$. Hence the diagram (3.2) and (theorem 3.1) gives, $(1) = SK_1(\text{gr}(D)) \simeq SK_1(D)$.

Case 2. Let $r_q = 0$. By (lemma 5.2), the group $\Gamma_D = \Gamma_F$. Thus if D is semiramified, then \overline{D} is a field and $\overline{D}/\overline{F} \simeq \Gamma_D/\Gamma_F = (1)$. Hence $D = F$ (cf. equation 2), and $SK_1(D) = SK_1(F) = (1)$. If D is totally ramified then, $D = F$ (cf. equation 3). Therefore $SK_1(D) = SK_1(F) = (1)$. ■

Remark 5.4 We keep notation of the above theorem (5.3). Suppose $r_q = 0$. By (proposition 8.59, [TW-15]), there are division algebras S and T with center F such that S is inertially split, T is tame and totally ramified over F , and $D \sim S \otimes_F T \in Br(F)$. As observed in the proof of the above theorem, for a tame totally ramified algebra T with $r_q = 0$, $\Gamma_T = \Gamma_F$ and $T \simeq F$. Thus, $D \sim S \otimes_F F \sim S$, i.e., the division algebra D is Brauer equivalent to an inertially split division algebra S . If \overline{S} is a field then, by (proposition 8.38, [TW-15]) S is semiramified over F . In this case, $SK_1(D) \simeq SK_1(S) = (1)$.

We now give some examples of fields satisfying conditions of the above theorem (5.3). We also give examples of division algebras over such fields.

Example 5.5 1. Consider a rational function field in one variable over p -adic field, $\mathbb{Q}_p(t)$ and let q be a prime distinct from p . By ([Se2]) the q -cohomological dimension of $\mathbb{Q}_p(t)$, $cd_q(\mathbb{Q}_p(t)) = 3$. We denote the completion of $\mathbb{Q}_p(t)$ with respect to a discrete valuation of rank 1 by $\widehat{\mathbb{Q}_p(t)}$. Thus the q -cohomological dimension of the completion $\widehat{\mathbb{Q}_p(t)}$, $cd_q(\widehat{\mathbb{Q}_p(t)}) \leq 3$. Moreover, as the valuation is discrete of rank 1 the q -rank of the valuation (as defined above), $r_q = 1$. We can obtain examples of division algebras over $\widehat{\mathbb{Q}_p(t)}$ as follows. By (proposition 1.16, [TW-15]) if D is a central division algebra over $\mathbb{Q}_p(t)$ such that a discrete valuation on $\mathbb{Q}_p(t)$ extends to D then, $D \otimes \widehat{\mathbb{Q}_p(t)}$ is a division ring.

2. Let k be a field of characteristic different from 2. Suppose that the q -cohomological dimension of k , $cd_q(k) = 2$. Consider the Laurent series field over k with the t -adic valuation, $F = k((t))$. Note that by (proposition 12, II.§4, [Se2]) $cd_q(F) = 3$. As the t -adic valuation is discrete of rank 1 the q -rank, $r_q = 1$ in this case. Let $u \in \mathcal{O}_F^*$ be a non-square unit and consider a quaternion algebra (u, t) over F . By (VI. 1.9, [L]), (u, t) is a division algebra over F . For a central simple algebra A over k we obtain a central simple algebra $A \otimes_F (u, t)$ over F .

There is a way to construct division algebra with center a Laurent series field by following example given in (§1.1.2, [TW-15]). Let k be a field as above and let $k \subseteq E$ be a finite cyclic Galois extension of degree m with Galois group $Gal(E/k) = \langle \sigma \rangle$. Let D be a finite-dimensional division algebra over E . Assume that σ extends to a k -algebra automorphism of D , we denote it again by σ . Consider

$$D((t, \sigma)) = \left\{ \sum_{i \geq n} d_i t^i : n \in \mathbb{Z}, d_i \in D \right\}, \text{ where } t \text{ is an indeterminate;}$$

with the multiplication defined by $t \cdot d = \sigma(d)t$ for $d \in D$. We identify D with the subring Dt^0 of $D((t, \sigma))$. For $z = \sum_{i \geq n} d_i t^i \in D((t, \sigma))$, define $supp(z) = \{i : d_i \neq 0\}$ and the valuation on $D((t, \sigma))$, $v_t(z) = \min(supp(z))$. Then $D((t, \sigma))$ is a division ring (cf. §1.1.2, [TW-15]).

By the Skolem-Noether theorem σ^m is an inner automorphism of D . Let $a \in D^*$ be such that $\sigma^m = \text{Inn}(a)$. Consider $x = at^m \in D((t, \sigma))$. Then for any $d \in D$,

$$\begin{aligned} x \cdot d &= at^m \cdot d \\ &= a\sigma^m(d)t^m \\ &= a(a^{-1}da)t^m \\ &= dat^m = dx. \end{aligned}$$

Hence $x = at^m \in D((t, \sigma))$ is an indeterminate which commutes with every element of D . The center of $D((t, \sigma))$ has the following description,

$$Z(D((t, \sigma))) = k((x)) = \left\{ \sum_{i \geq n} d_i (at^m)^i : d_i \in k \text{ and } n \in \mathbb{Z} \right\}.$$

We also have $\deg D((t, \sigma)) = m \cdot \deg D$. Consider $E = k$ and $\sigma = Id_k$. In this case the center $Z(D((t, \sigma))) = k((x)) = k((t))$ is Henselian and the restriction of the valuation v_t to $Z(D((t, \sigma)))$ is discrete of rank 1. As a result the q -rank, $r_q = 1$. By (II.§4, proposition 12, [Se2]) if $cd_q(k) = 2$ then, $cd_q(k((x))) = 3$.

3. Let k be an arbitrary field of q -cohomological dimension, $cd_q(k) = 1$. Let q be a prime number distinct from the characteristic of k and let $q^s \geq 2$ for some natural number s . Assume that k contains a primitive q^s -th root of unity ω . Define the twice iterated Laurent series field over k , $F = k((x))((y))$ with the (x, y) -adic valuation. The q -cohomological dimension of F , $cd_q(F) = 3$ (cf. proposition 12, II.§4, [Se2]), and the field F is Henselian for the (x, y) -adic valuation, with residue field k and the value group \mathbb{Z}^2 . In this case the q -rank of $\Gamma_F = \mathbb{Z}^2$, $r_q = 2$. The symbol algebra $(x, y)_{\omega, q^s}$ over F is a central division algebra of degree and exponent q^s , and it is a tame totally ramified division algebra with the value group $(\frac{1}{n}\mathbb{Z})^2$ (cf. §9.1.3, [TW-15]).

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